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SHORTER COMMUNICATIONS

NATURAL CONVECTION IN A POROUS LAYER SATURATED WITH A COMPRESSIBLE IDEAL GAS

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NOMENCLATURE

C_1, C_2,	constants used to fit ρ_0 with a straight line;
C_I,	constant of integration;
g,	acceleration due to gravity [m s^{-2}];
Ga,	Gay–Lussac number $(1/\beta\Delta T)$;
K,	permeability [m^2];
l,	thickness of the porous layer [m];
M,	molecular weight of the fluid;
P,	dimensionless pressure;
P_0,	reference pressure [kg ms^{-2}];
R,	universal gas constant [j mole^{-1} K^{-1}];
Ra,	Rayleigh number $\left[\dfrac{K(\rho c)_f\,\rho_0 g l\beta\Delta T}{\mu\lambda_0}\right]$;
R_T,	dimensionless lower wall temperature;
t,	dimensionless time;
T,	dimensionless temperature;
T_{amb},	ambient temperature (300 K)
u, v,	dimensionless horizontal and vertical velocity components;
x, y,	dimensionless horizontal and vertical coordinates.

Greek symbols

β,	coefficient of thermal expansion [K^{-1}];
ε,	porosity;
λ_0,	thermal conductivity of the porous medium [W m^{-1} K^{-1}];
μ,	fluid viscosity [kg ms^{-1}];
ρ,	dimensionless fluid density;
ρ_0,	reference fluid density [kg m^{-3}];
(ρc),	heat capacity per unit volume of the porous medium [J m^{-3} K^{-1}];
$(\rho c)_f$,	heat capacity per unit volume of fluid (J m^{-3} K^{-1}].

INTRODUCTION

FOR A POROUS medium bounded by two isothermal impermeable planes, Lapwood [1] predicted the criteria for the onset of convection. The transition Rayleigh number that he derived ($4\pi^2$) has been verified experimentally [2]. However, the mathematical formulation is simplified by the Boussinesq approximation which states that the fluid density differences are to be considered in the buoyancy terms only. Thus, a simple stream function can be defined.

In this report, we present a different formulation which takes into account of all density variations for an ideal gas. The critical conditions for the transitions from conduction to convection are determined by a linear theory and agree remarkably well with previous work [3, 4].

GOVERNING EQUATIONS

Assuming that the porous layer is saturated with an ideal gas and considering the fluid density as variable, the system of equations which represents the phenomena is:

$$\varepsilon\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\mathbf{V} = 0 \tag{1}$$

$$V = -\frac{K}{\mu}[\nabla P - \rho\mathbf{g}] \tag{2}$$

$$(\rho c)\frac{\partial T}{\partial t} + (\rho c)_f\,\mathbf{V}\cdot\nabla T = \Lambda_0\nabla^2 T \tag{3}$$

$$P = \frac{\rho R T}{M}. \tag{4}$$

Equations (1)–(4) are rendered dimensionless with the following reference parameters: ρ_0 for the fluid density, $(\rho c)l^2/\lambda_0$ for time, $\lambda_0/(\rho c)_f\,l$ for the velocities, T_{amb} for the temperature, l for length and P_0 for pressure. Inserting the ideal gas law into Darcy's equation, the dimensionless, two dimensional system becomes:

$$\varepsilon\frac{\partial\rho}{\partial t} + \frac{(\rho c)}{(\rho c)_f}\left[u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right] = 0 \tag{5}$$

$$u = -\frac{K(\rho c)_f P_0}{\mu\Lambda_0}\left[\rho\frac{\partial T}{\partial x} + T\frac{\partial\rho}{\partial x}\right] = 0 \tag{6}$$

$$v = \frac{K(\rho c)_f P_0}{\mu\Lambda_0}\left[\rho\frac{\partial T}{\partial y} + T\frac{\partial\rho}{\partial y}\right] - RaGa\rho \tag{7}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}. \tag{8}$$

Three dimensionless parameters appear in the equations: $(\rho c)/(\rho c)_f$, $K(\rho c)_f P_0/\mu\lambda_0$ and $Ra\,Ga$. When dealing with natural convection, the pressure gradients are caused by fluid density differences and thus we can assume that $P_0 = \rho g l$. Equations (6)–(7) can then be written:

$$u = -Ra\,Ga\left[\rho\frac{\partial T}{\partial x} + T\frac{\partial\rho}{\partial x}\right] \tag{6'}$$

$$v = -Ra\,Ga\left[\rho\frac{\partial T}{\partial y} + T\frac{\partial\rho}{\partial y} + \rho\right]. \tag{7'}$$

The boundary and initial conditions in dimensionless form are:

$$T = R_T;\quad v = 0\quad \text{for } y = 0$$

$$T = 1;\quad v = 0\quad \text{for } y = 1$$

$$\frac{\partial T}{\partial x} = 0; \quad u = 0 \quad \text{for } x = 0 \text{ and } x = 1 \quad (9)$$

$$\rho = 1; \quad T = 1 \quad \text{for } t = 0.$$

With the above formulation, the *Ra Ga* number instead of the Rayleigh number appears to be the critical parameter. Since the isothermal compressibility β is equal to $1/T$ for an ideal gas, a simple equation will enable us to deduce one from the other.

LINEAR STABILITY ANALYSIS

The critical conditions for the transition from a motionless, pure conduction regime to steady convection are determined by a linear theory.

The motionless reference state is independent of the lateral coordinate x and is defined by:

$$u_0 = v_0 = 0$$

$$T_0 = R_T + (1 - R_T)y.$$

Substitution into (7′) leads to
where C_I is a constant of integration determined by

$$\int_{y=0}^{y=1} \rho_0 \, \mathrm{d}y = 1.$$

In order to facilitate the method of solution that follows we approximate the curve ρ_0 by a polynomial $\rho_0 = C_1 y + C_2$. Notice that for $R_T = 2$, the reference density distribution becomes $\rho_0 = 1$.

Introducing temperature θ, velocity u', v' and density perturbations such that:

$$\begin{aligned} T &= T_0 + \theta \\ u &= u_0 + u' \\ v &= v_0 + v' \\ \rho &= \rho_0 + \rho \end{aligned} \quad (10)$$

into the system of equations (5, 6′, 7′, 8) and neglecting all second terms we obtain:

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = - Ra\, Ga \left[T_0 \frac{\partial \rho'}{\partial y} + \rho_0 \frac{\partial \theta}{\partial y} + \frac{\mathrm{d}\rho_0}{\mathrm{d}y}\theta + \frac{\mathrm{d}T_0}{\mathrm{d}y}\rho' + \rho' \right] \quad (11)$$

$$\rho_0 \left[\rho_0 \left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right) + T_0 \left(\frac{\partial^2 \rho'}{\partial x^2} + \frac{\partial^2 \rho'}{\partial y^2} \right) + \left(2\frac{\mathrm{d}T_0}{\mathrm{d}y} + 1 \right) \frac{\partial \rho'}{\partial y} + 2 \frac{\mathrm{d}\rho_0}{\mathrm{d}y} \frac{\partial \theta}{\partial y} \right] = - \frac{\mathrm{d}\rho_0}{\mathrm{d}y} \left[\left(\frac{\mathrm{d}T_0}{\mathrm{d}y} + 1 \right) \rho' + \frac{\mathrm{d}\rho_0}{\mathrm{d}y}\theta + \rho_0 \frac{\partial \theta}{\partial y} + T_0 \frac{\partial \rho'}{\partial y} \right]. \quad (12)$$

Assuming solutions, for the two equations above, of the form:

$$\rho' = \rho''(y)\, e^{i\alpha x} \qquad \theta = \theta''(y)\, e^{i\alpha x}$$

and inserting them into the equations we obtain two differential equations of the form

$$\frac{\mathrm{d}^2\theta''}{\mathrm{d}y^2} - (z_1 y + z_2)\frac{\mathrm{d}\theta''}{\mathrm{d}y} - z_3\theta'' = z_4 \rho'' + (z_5 + z_6)\frac{\mathrm{d}\rho''}{\mathrm{d}y} \quad (13)$$

$$[p_1 y^2 + p_2 y + p_3]\frac{\mathrm{d}^2\theta''}{\mathrm{d}y^2} + [p_4 y + p_5]\frac{\mathrm{d}\theta''}{\mathrm{d}y} - [p_6 y^2 + p_7 y + p_8]\theta'' + [p_9 y^2 + p_{10} y + p_{11}]\frac{\mathrm{d}^2\rho''}{\mathrm{d}y^2} + [p_{12} y + p_{13}]\frac{\mathrm{d}\rho''}{\mathrm{d}y} + [p_{14} y^2 + p_{15} y + p_{16}]\rho'' = 0 \quad (14)$$

where $z_{1\to 6}$ and $p_{1\to 16}$ are known constants.

The two differential equations are solved by series:

$$\theta'' = \sum_{n=0}^{r} a_n y^n \qquad \rho'' = \sum_{n=0}^{r} b_n y^n$$

The boundary conditions for the perturbations are:

$$\theta'' = 0; \quad v' = 0 \quad \text{for } y = 0 \quad \text{and } y = 1.$$

The linearized expression for v', into which the series solutions for θ'' and ρ'' are substituted to satisfy the velocity conditions is

$$v' = - Ra\, Ga \left[\left(\frac{\mathrm{d}T_0}{\mathrm{d}y} + 1 \right) \rho'' + \theta'' \frac{\mathrm{d}\rho_0}{\mathrm{d}y} + \rho_0 \frac{\mathrm{d}\theta''}{\mathrm{d}y} + T_0 \frac{\mathrm{d}\rho''}{\mathrm{d}y} \right].$$

A matrix of coefficients $a_0, a_1, \ldots, a, b_0, b_1, \ldots, b_n$ can thus be constructed from the recurrence relations and from the boundary conditions. Its determinant equals zero for the critical *Ra Ga* number.

The first 26 terms of each series are sufficient to give a precise value (error less than 1%) for $Ra\, Ga_c$. Table 1 shows, for different R_T, the critical *Ra Ga* number. Using the average temperature in the layer $\bar{T} = T_p + T_{amb}/2$ to calculate β, the critical Rayleigh number is deduced.

With the exception of $R_T = 1.5$, the critical Rayleigh numbers derived are slightly lower than $4\pi^2$; the moderate density differences accounted for in all terms are responsible for this. As the temperature difference across the layer increases, the density gradients become more important and the critical Rayleigh number decreases. These results agree well with those of Epherre, Combarnous and Klarsfeld [3] who, using a different formulation considered both the density and the fluid viscosity variations in an anisotropic medium. For $R_T = 1.5$, the value of Ra_c is higher than $4\pi^2$. This is due to the fact that the mixed mean temperature is higher than the average temperature once convective movement appears. Thus, the real Ra_c numbers are a bit below our tabulated values.

The solution of the two ordinary differential equations by series allows us to use a polynomial of any degree to approximate $\rho_0(y)$. As R_T increases, the curve $\rho_0(y)$ is less well represented by a straight line.

CONCLUSIONS

A model for natural convection in a porous layer saturated with an ideal gas is presented. The fluid density differences are accounted for in all terms. The linear stability analysis leads us to two coupled ordinary differential equations which are solved by series. Due to the moderate density differences, the critical Rayleigh number obtained is slightly below $4\pi^2$. As the imposed temperature gradient across the layer is increased, the critical Rayleigh number decreases.

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Table 1

R_T	ρ_0	C_1	C_2	$Ra\ Ga_c$	Ra
1.5	$\left[\dfrac{1.25}{1.5-0.5y}\right]^{-1}$	−0.4	1.2	101.68	40.672
2	1	0	1	58.9267	39.284
2.5	$\left[\dfrac{1.6751}{2.5-1.5y}\right]^{1/3}$	0.3031	0.8516	44.32	37.9886
3	$\left[\dfrac{1.86605}{3-2y}\right]^{1/2}$	0.5477	0.7340	37.2	37.2

$R_T = \dfrac{T_p}{T_{amb}}$; $T_{amb} = 300\,K$; $S = 3.15$.

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ANALYSIS OF BATCH SCRAPED SURFACE HEAT EXCHANGE

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NOMENCLATURE

- A, imperfect mixing parameter: ratio of that fraction of the scraped layer which is well mixed with the core: $vm/(vm + M)$;
- B', perfect scraping parameter: $\dfrac{4}{R}\dfrac{\sqrt{\alpha\theta_c}}{\pi}$;
- B, parameter including external heat transfer resistance $\dfrac{U}{h_f}\dfrac{4}{R}\dfrac{\sqrt{\alpha\theta_c}}{\pi}$;
- C_p, heat capacity [J/gK];
- h_e, heat transfer coefficient between external fluid (Bath) and wall;
- h_f, theoretical scraped surface heat transfer coefficient: $2k/\sqrt{\pi\alpha\theta_c}$ [W/m^{2}K];
- k, thermal conductivity of mixture of fluid [W/mK];
- k_w, thermal conductivity of heat exchange wall 16.5 [W/mK];
- l, thickness of heat exchanger wall (1.01 mm);
- m, mass of scraped layer per scraping cycle [g];
- M, mass of central (mixed) core [g];
- n, total number of scrapings (θ/θ_c);
- R, radius of cylindrical scraped surface heat exchanger;
- U, overall heat transfer coefficient between external bath and scraped surface heat exchanger wall:

$$\frac{1}{U} = \frac{1}{h_e} + \frac{l}{k_w} + \frac{\delta_i}{k} + \frac{1}{h_f}\ [\mathrm{W/m^2K}];$$

- T, average temperature of well mixed core material [°K];
- T_w, wall or bath temperature [K];
- T_0, initial temperature of mixture system [K].

Greek symbols

- α, thermal diffusivity of mixture of fluid [m^2/s];
- δ_i, thickness of residual film left by the scraper blade during imperfect scraping [m];
- δ, scraped film thickness (scraper blade width) [m];
- v, ratio of mass of intermediate mixing layer to scraped layer;
- θ, total time $(n\theta_c)$ [s];
- θ_c, time between scrapings [s].

INTRODUCTION

WE HAVE made an analysis [1,2] of scraped surface heat exchange [3–12] in a cylindrical batch vessel which extends the traditional suggestion of Houlton [13] to include the effects of imperfect scraping and incomplete mixing between the material removed by the scraping and the main body of the heated material. There is current interest in the process [14] because the increase in heat transfer by scraping can be obtained for less mechanical energy than that required by stirring or agitation with baffles when the material has a thermal conductivity less than 1 W/mK, even if it is not very viscous [15].

When the process is regarded as a sequence of contacts with the wall and subsequent perfect mixing with a well mixed core, one obtains the average dimensionless temperature of the layer for the nth scraping as

$$\frac{T_a^{n+1} - T^n}{T_w - T^n} = \frac{2k\sqrt{\theta_c}}{\sqrt{(\pi\alpha)}\rho\delta C_p}. \tag{1}$$